Math 141 Section 2.2
Some possible results of Gauss-Jordan Elimination

Usually:

equations = # unknowns $\Rightarrow$ 1 solution
equations > # unknowns $\Rightarrow$ 0 solution
equations < # unknowns $\Rightarrow$ ∞ solutions
There are sometimes exceptions.

Situation	Typical number of solutions	Example	Row reduced augmented matrix	In equation form	So solution is	Solutions
# equations = # unknowns	1	$x + 2y = 5$ $3x + 4y = 6$	$\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cc\|c} 1 & 0 & -4 \\ 0 & 1 & 9/2 \end{array}\right]$	$x = -4$ $y = 9/2$	$x = -4$ $y = 9/2$	Only $(-4, 9/2)$
# equations > # unknowns	0	$x + 2y = 5$ $3x + 4y = 6$ $7x + 8y = 9$	$\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 3 & 4 & 6 \\ 7 & 8 & 9 \end{array}\right] \to \left[\begin{array}{cc\|c} 1 & 0 & -4 \\ 0 & 1 & 9/2 \\ 0 & 0 & 1 \end{array}\right]$	$x = -4$ $y = 9/2$ $0 = 1$	No solution	None
# equations < # unknowns	∞	$x + 2y + 3z = 6$ $4x + 5y + 6z = 12$	$\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 12 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \end{array}\right]$	$x - z = -2$ $y + 2z = 4$	$x = -2 + z$ $y = 4 - 2z$ $z = z$	$(-2, 4, 0)$ or $(-1, 2, 1)$ or $(0, 0, 2)$ ...
# equations > # unknowns	0	$x + 2y = 5$ $3x + 4y = 6$ $7x + 8y = 8$	$\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 3 & 4 & 6 \\ 7 & 8 & 8 \end{array}\right] \to \left[\begin{array}{cc\|c} 1 & 0 & -4 \\ 0 & 1 & 9/2 \\ 0 & 0 & 0 \end{array}\right]$	$x = -4$ $y = 9/2$ $0 = 0$	$x = -4$ $y = 9/2$	Only $(-4, 9/2)$
# equations = # unknowns	1	$x + 2y + 3z = 6$ $4x + 5y + 6z = 12$ $7x + 8y + 9z = 18$	$\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 12 \\ 7 & 8 & 9 & 18 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$	$x - z = -2$ $y + 2z = 4$ $0 = 0$	$x = -2 + z$ $y = 4 - 2z$ $z = z$	$(-2, 4, 0)$ or $(-1, 2, 1)$ or $(0, 0, 2)$ ...
# equations = # unknowns	1	$x + 2y + 3z = 6$ $4x + 5y + 6z = 12$ $7x + 8y + 9z = 19$	$\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 12 \\ 7 & 8 & 9 & 19 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$	$x - z = -2$ $y + 2z = 4$ $0 = 1$	No solution	None

Other possible results of Gauss-Jordan Elimination

(we use $x_1, x_2, \ldots$ rather than $x, y, \ldots$)

Reduced row echelon matrix	Solutions (regardless of RHS)	Free variable in solution	Solution
$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$	1		$x_1 = 2$ $x_2 = 3$ $x_3 = 4$
$\left[\begin{array}{cccc\|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 4 & 7 \end{array}\right]$	∞	x_4	$x_1 = 5 - 2x_4$ $x_2 = 6 - 3x_4$ $x_3 = 7 - 4x_4$ $x_4 = x_4$
$\left[\begin{array}{cccc\|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$	∞	x_3	$x_1 = 4 - 2x_3$ $x_2 = 5 - 3x_3$ $x_3 = x_3$ $x_4 = 6$
$\left[\begin{array}{ccccc\|c} 1 & 2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 & 7 \end{array}\right]$	∞	$x_2,\ x_5$	$x_1 = 5 - 2x_2 - 3x_5$ $x_2 = x_2$ $x_3 = 6$ $x_4 = 7 - 4x_5$ $x_5 = x_5$
$\left[\begin{array}{cc\|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & a \end{array}\right]$	1 if $a = 0$ 0 if $a \neq 0$		$x_1 = 2$ $x_2 = 3$
$\left[\begin{array}{ccc\|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & a \end{array}\right]$	∞ if $a = 0$ 0 if $a \neq 0$	x_3 if there is a solution	$x_1 = 4 - 2x_3$ $x_2 = 5 - 3x_3$ $x_3 = x_3$
$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & a \end{array}\right]$	1 if $a = 0$ 0 if $a \neq 0$		$x_1 = 2$ $x_2 = 3$ $x_3 = 4$
$\left[\begin{array}{cccc\|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & a \end{array}\right]$	∞ if $a = 0$ 0 if $a \neq 0$	x_4 if there is a solution	$x_1 = 4 - 2x_4$ $x_2 = 5 - 3x_4$ $x_3 = 6$ $x_4 = x_4$
$\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \end{array}\right]$	∞ if $a = b = 0$ 0 otherwise	$x_2,\ x_3$ if there is a solution	$x_1 = 4 - 2x_2 - 3x_3$